International Journal of Theoretical and Applied Science 3(1): 49-50(2011) ISSN : 0975-1718

Generalized Ishikawa Iterates converging to a common fixed point of three mappings

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(Received 11 Jan., 2011, Accepted 12 Feb., 2011)

ABSTRACT : In this paper it is shown that if the generalized sequence of Ishikawa iterates associated with three mappings converges, its limit is the common fixed point of these mappings. This results extends and generalizes the corresponding results of Ciric *et al.* [2], Naimpally and Singh [4] and Rhoades [6].

2000 Mathematics Subject Classification : 54H25

Keywords : Covex metric spaces, Generalized Ishikawa iterative scheme, common fixed point.

I. INTRODUCTION

W. Takahashi [8] introduced a new concept of convexity in metric spaces and generalized some important fixed point theorems previously proved for Banach spaces and named them convex metric spaces.

Definition 1. Let (X, d) be a metric space and I = [0, 1] the closed unit interval. A Takahashi convex structure on X is a function $W : X \times X \times I \to X$ which has the property that for every $x, y \in X$ and $t \in I$

$$d(z,W(x, y, t)) \leq t\, d(z, x) + (1 - t)d(z, y)$$

for every $z \in X$.

If (X, d) is equipped with a Takahashi convex structure, then X is called a convex space. A Banach space, or any convex subset of it is a convex metric space with $W(x, y, t) = tx + (1 - t)x$.

Definition 2. Let X be a convex metric space. A nonempty subset A of X is said to be convex if $W(x, y, t) \in A$ whenever $(x, y, t) \in A \times A \times [0, 1]$.

The Ishikawa iterative scheme [3] which was first used to establish the strong convergence for a pseudocontractive selfmapping of a convex compact subset of a Hilbert space was soon used to establish the strong convergence of its iterates for some contractive type mappings in Hilbert spaces and then in more general normed linear spaces.

Ciric *et al.* [2] extended a result of Naimpally and Singh [4] involving Ishikawa interactive scheme to convex metric spaces which goes like this :

Theorem 1. *Let C be a nonempty closed convex subset of a convex metric space X and let S, T : X → X be self mappings satisfying (A) for all x, y in C.*

$$d(Sx, Ty) \leq h[d(x, y) + d(x, Ty) + d(y, Sx)] \quad ...(A)$$

Suppose that $\{x_n\}$ is Ishikawa type iterative scheme associated with S and T, defined by

$$x_0 \in C,\ y_n = W(Sx_n, x_n, \beta_n),\ x_{n+1} = W(Ty_n, x_n, \alpha_n),\ n \leq 0$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy* $0 \leq \alpha_n, \beta_n \leq 1$ and $\{\alpha_n\}$ is bounded away from zero. If $\{x_n\}$ converges to some point $p \in C$, *then p is the common fixed point of S and T.*

Our purpose is to use generalized Ishikawa iterative scheme (three-step iteration method) [9], which is more general than Ishikawa iterations, to extend the above mentioned result for three mappings.

Definition 2. For a given $x_0 \in C \subset X$ and self mappings S, T and $U : X \to X$ define sequence $\{x_n\}$ by

$$z_n = W(Sx_n, x_n, c_n), \quad n \geq 0,\ c_n \in [0, 1]$$
$$y_n = W(Tz_n, x_n, b_n), \quad n \geq 0,\ b_n \in [0, 1]$$
$$x_{n+1} = W(Uy_n, x_n, a_n), \quad n \geq 0,\ a_n \in [0, 1]$$

This is called generalized Ishikawa iterative scheme.

For $c_n = 0$ the above iterative scheme reduces to ;

For a given $x_0 \in C \subset X$ and self mappings $T, U : X \to$ define sequence $\{x_n\}$ by

$$y_n - W(Tx_n, x_n, b_n), \quad n \geq 0,\ b_n \in [0,1]$$
$$x_{n+1} = W(Uy_n, x_n, a_n), \quad n \geq 0,\ a_n \in [0,1]$$

which is Ishikawa iterative scheme.

II. MAIN RESULT

Theorem 1. *Let C be a non-empty closed convex subsed of a convex metric space. Let S, T and U : X → X be self mappings satisfying*

$$d(Sx, Uy) \leq h_1 [d(x, z) + d(z, Sx) + d(x, Uy)] \quad ...(A_1)$$

and $d(Uy, Tz) \leq h_2 [d(y, x) + d(x, Uy) + d(y, Tz)] \quad ...(A_2)$

for all $x, y, z \in C$, where $0 < h_1, h_2 < 1$ and let $\{x_n]\}$ *be generalized Ishikawa iterative scheme associated with S, T and U i.e*

1. $z_n = W(Sx_n, x_n, c_n), \quad n \geq 0,\ c_n \in [0, 1]$
2. $y_n = W(Tz_n, x_n, b_n), \quad n \geq 0,\ b_n \in [0, 1]$
3. $x_{n+1} = W(Uy_n, x_n, a_n), \quad n \geq 0,\ a_n \in [0, 1]$

and $\lim_{n\to\infty} a_n = 0$. If $\{x_n\}$ *converges to some point* $p \in C$, *then p is the common fixed point of S, T and U.*

Proof. By the definition of convex metric space we deduce that

$d(x, y) \le d[x, W(x, y, t) + d[W(x, y, t), y]$
$\le t\, d(x, x) + (1 - t)\, d(x, y) + t d(y, x) + (1 - t) d(y, y)$
$= (1 - t)\, d(x, y) + t d(x, y)$
$= d(x, y)$

which implies that

$d[x, W(x, y, t)] = (1 - t)\, d(x, y)\ ;\ d[y, W(x, y, t)] = t\, d(x, y)$

From (3), we have

$d(x_n, x_{n+1}) = d[x_n, W(Uy_n, x_n, a_n)] = a_n\, d(x_n, Uy_n).$

Since $x_n \to p$, $d(x_n, x_{n+1}) \to 0$. Moreover $\lim_{n\to\infty} a_n = 0$ following that

$d(x_n, Uy_n) \to 0$ as $n \to \infty$ showing $Uy_n \to p$ ...2.1

From (1)

$d(x_n, z_n) = d[x_n, W(Sx_n, x_n, c_n)] = c_n\, d(x_n, Sx_n)$ ...2.2

$d(z_n, Sx_n) = d[W(Sx_n, x_n, c_n), Sx_n] = (1 - c_n)\, d(x_n, Sx_n)$

Using condition (A_1), we get

$d(Sx_n, Uy_n) \le h_1 [d(x_n, z_n) + d(z_n, \delta x_n) + d(x_n, Uy_n)]$
$= h_1 [c_n\, d((x_n, Sx_n) + (1 - c_n) d(x_n, Sx_n) + d(x_n, Uy_n)]$
$= h_1 [d((x_n, Sx_n) \mid d(x_n, Uy_n)]$

Since $d(x_n, Sx_n) \le d((x_n, Uy_n) + d(Sx_n, Uy_n)$, we have

$d(Sx_n, Uy_n) \le h_1 [d(x_n, Uy_n) + d(Sx_n, Uy_n) + d(x_n, Uy_n)]$
$= h_1[2d(x_n, Uy_n) + d(Sx_n, Uy_n)]$

Hence,

$$d(Sx_n, Uy_n) \le \frac{2h_1}{1-h_1} d((x_n, Uy_n)$$

Taking the limit as $n\to\infty$ and using equation 2.1 we get,

$\lim_{n\to\infty} d(Sx_n, Uy_n) = 0$

Since $Uy_n \to p$, above equation implies that $Sx_n \to p$.

Also from 2.2

$d(x_n, z_n) = c_n\, d(x_n, Sx_n)$

Hence $z_n \to p$

From condition (A_1), we have

$d(Sx_n, Up) \le h_1[d((z_n, z_n) + d(zn, Sx_n) + d(x_n, Up)]$

Taking the limit as $n \to \infty$ we get

$d(p, Up) \le h_1, d((Up)$

which implies that $d(p, Up) = 0$, since $h_1 < 1$. Thus $Up = p$

Similarly from (A_1)

$d(Sp, Uy_n) \le h_1[d((p, x_n) + d(z_n, S_p) + d(p, Uy_n)]$

Taking the limit as $n\to\infty$ we obtain

$d(Sp, p) \le h_1 d((p, Sp)$

Thus $Sp - p$, since $h_1 < 1$.

Again from (2)

$d(x_n, y_n) = d[x_n, W(Tz_n, x_n, b_n)] = b_n d(x_n, Tz_n)$ ...2.3

$d(y_n, Tz_n) = d[W(Tz_n, x_n, b_n)\ Tz_n] = (1 - b_n) d(x_n, Sx_n)$

From condition (A_2)

$d(Uy_n, Tz_n) \le h_2[d(x_n, y_n) + d(x_n, Uy_n) + d(y_n, Tz_n)]$
$= h_2[b_n\, d((x_n, Tz_n) + (1 - b_n) d(x_n, Tz_n) + d(x_n, Uy_n)]$
$= h_2[d(x_n, Tz_n) + d(x_n, Uy_n)]$
$\le h_2[d(x_n, Uy_n) + d(Uy_n, Tz_n) + d(x_n, Uy_n)]$
since $d(x_n, Tz_n) \le d(x_n, Uy_n) + d(Uy_n, Tz_n)\}$
$= h_2[2d((x_n, Uy_n) + d(Uy_n, Tx_n)]$

Hence, $d(Uy_n, Tz_n) \le \dfrac{2h_2}{1-h_2} d((x_n, Uy_n)$

Taking the limit as $n\to\infty$ and using 2.1 we get,

$\lim_{n\to\infty} d(Uy_n\, Tz_n) = 0$

showing that $Tz_n \to p$ which along with 2.3 show $y_n \to p$

From (A_2)

$d(Uy_n Tp) \le h_2[d(x_n, y_n) + d(x_n, Uy_n) + d(y_n, Tp)]$
$d(p, Tp) \le h_2[d(p, p) + d(p, p) + d(p, Tp)]$

implying that

$d(p, Tp) = 0$, since $h_2 < 1$

Therefore $Tp = $ p.

Hence $Sp = Tp = Up = p$. This completes the proof.

Corollary 1. Let X be a normed linear space and C and C be a closed convex subset of X. Let S, T and U be three mapings satisfying A_1 and A_2 and $\{x_n\}$ be the generalized sequence of Ishikawa iterative scheme associated with S, T and U ; for $x_0 \in C$.

$z_n = (1 - c_n)x_n + c_n Sx_n, \quad n \ge 0,\ c_n \in [0,1]$
$y_n = (1 - b_n)x_n + b_n Tz_n, \quad n \ge 0,\ b_n \in [0,1]$
$x_{n+1} = (1 - a_n)x_n + a_n Uy_n, \quad n \ge 0,\ a_n \in [0,1]$

If $\lim_{n\to\infty} a_n = 0$ and $\{x_n\}$ converges to p, then p is the common fixed point of S, T and U.

Remark. For $c_n = 0$, lthe three step iterative scheme reduces to Ishikawa iterative scheme, thus giving the corresponding results of Ciric *et al.* [2], Naimpally and Singh [4] and Rhoades [6].

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